

Gauss-Lucas Theorems for Entire Functions on C^M

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Abstract. A Gauss-Lucas theorem is proved for multivariate entire functions, using a natural notion of separate convexity to obtain sharp results. Previous work in this area is mostly restricted to univariate entire functions (of genus no greater than one unless “realness” assumptions are made). The present work applies to multivariate entire functions whose sections can be written as a monomial times a canonical product of arbitrary genus. Essential use is made of the Levy-Steinitz theorem for conditionally convergent vector series, a result generalizing Riemann’s well known theorem for conditionally convergent real number series.

Key words and phrases. Gauss-Lucas theorem, convex hull, entire function, multivariate function, stable polynomial, Levy-Steinitz theorem, conditionally convergent series.

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1 Introduction

Let $f(z)$ be a non-constant univariate polynomial and let $f_{,1}(z)$ stand for the complex derivative $\frac{d}{dz}f(z)$. The classical Gauss-Lucas theorem is the set relation

$$f_{,1}^{(-1)}(0) \subset H(f^{(-1)}(0)) , \quad (1)$$

where $f_{,1}^{(-1)}(0) \equiv \{z : f_{,1}(z) = 0\}$ and $H(f^{(-1)}(0))$ is the convex hull of the roots of $f(z)$ in the complex plane C . It is of interest to extend this elegant result to entire functions on C^M .

The desired Gauss-Lucas relation for non-constant entire univariate functions $f(z)$ is the set inclusion

$$f_{,1}^{(-1)}(0) \subset \bar{H}(f^{(-1)}(0)) , \quad (2)$$

where $\bar{H}(f^{(-1)}(0))$ is the closure of $H(f^{(-1)}(0))$ in the complex plane. (Closure is taken since $f^{(-1)}(0)$ is not necessarily finite.) Theorem 1 (in Section 2) proves this relation under three assumptions:

- a) that $f(z) = z^q g(z)$, where q is a non-negative integer and $g(z)$ is a canonical product of finite genus p with $g(0)=1$.
- b) that $g(z)$ has only finitely many zeros in any bounded subset of C .
- c) that the non-zero roots $(\gamma_n : n \geq 1)$ of $g(z)$ satisfy

$$\sum_{x_n \geq 0}^{\infty} x_n = \infty , \quad (3)$$

where $(x_n : n \geq 1)$ is either the sequence $Re(\gamma_n^{-1})$, or the sequence $Re(\gamma_n^{-2}), \dots$, or the sequence $Re(\gamma_n^{-p})$, or any one of the remaining sequences $Re(-\gamma_n^{-r}), Im(\gamma_n^{-r}), Im(-\gamma_n^{-r})$ for $1 \leq r \leq p$.

Previous work extending the classical Gauss-Lucas theorem has been limited to entire functions of genus no greater than one, unless “realness” assumptions are made. (See e.g. [2], [3], [7], or [8].) Here, these restrictions are removed by appealing to the Levy-Steinitz theorem for vector series, which generalizes Riemann’s theorem that a conditionally convergent series of real numbers (convergent but not absolutely convergent) can be rearranged to sum to any real number.

To formulate the Gauss-Lucas relation for *multivariate* entire functions $f(z) = f(z_1, \dots, z_M)$, the notation $f_{,m}(z) \equiv \frac{\partial}{\partial z_m} f(z)$ and

$$f(w, z^{(m)}) \equiv f(z_1, \dots, z_{m-1}, w, z_{m+1}, \dots, z_M)$$

is convenient, where $z^{(m)} \equiv (z_1, \dots, z_{m-1}, z_{m+1}, \dots, z_M)$ is in C^{M-1} . It is then possible to define $f(z)$ as being an entire function on C^M if $g(w) \equiv f(w, z^{(m)})$ is an entire function on C for all $z^{(m)} \in C^{M-1}$. (See Section 3.)

The obvious notion of convexity in C^M , inherited by identifying C^M with R^{2M} , can be extended in a natural way to yield sharper results at no cost to simplicity. Defining the

projection operator $P_m(z) = z_m$, a set $K \subset C^M$ is called *separately convex in C^M* if the section

$$K(z^{(m)}) \equiv P_m(\{y : y \in K, y^{(m)} = z^{(m)}\})$$

is a convex subset of C for all $m \in \{1, \dots, M\}$ and all $z^{(m)} \in C^{M-1}$. The class of all separately convex subsets of C^M is clearly closed under intersection. Thus the convex hull of a set $A \subset C^M$ with respect to this notion of convexity may be identified as the smallest set containing A and separately convex in C^M . It is denoted $H_2(A)$.

Assume the partial derivative $f_{,m}(z)$ of $f(z)$ is not identically zero in z . The desired Gauss-Lucas relation for multivariate entire functions $f(z)$ is the set inclusion

$$f_{,m}^{(-1)}(0) \subset \bar{H}_2(f^{(-1)}(0)) , \quad (4)$$

where $\bar{H}_2(f^{(-1)}(0))$ is the closure of $H_2(f^{(-1)}(0))$ in C^M . Theorem 2 (in Section 3) establishes (4) for a general class of multivariate entire functions of finite genus. (A multivariate entire function $f(z)$ is said to have finite genus if the univariate function $g(w) \equiv f(w, z^{(m)})$ has finite genus for all $m \in \{1, \dots, M\}$ and all $z^{(m)} \in C^{M-1}$.)

Any subset of C^M which is convex in the usual sense is also separately convex. Letting $H(f^{(-1)}(0))$ be the usual convex hull of $f^{(-1)}(0)$, it follows that

$$H_2(f^{(-1)}(0)) \subset H(f^{(-1)}(0)) , \quad (5)$$

whence the set relation (4) is sharper than the corresponding relation using the standard notion of convexity in C^M . It is shown in Section 3 that the sharper relation (4) implies related recent results about multivariate stable polynomials. (See [10] for an exposition of these results, wherein a polynomial on C^M is termed stable if it has no roots z with $\text{Im}(z_m) > 0$ for all m .) If f is a multivariate polynomial then closure need not be taken in (4), yielding

$$f_{,m}^{(-1)}(0) \subset H_2(f^{(-1)}(0)) . \quad (6)$$

(See [5, Theorem 1] for this result and further comments on separately convex subsets of C^M .)

2 Univariate entire functions

This section relates the Gauss-Lucas theorem for univariate entire functions to the Levy-Steinitz generalization of Riemann's theorem for conditionally convergent real series.

An entire function $f(z)$ (with non-zero roots $\gamma = (\gamma_n : n \geq 1)$ and a root at the origin of multiplicity q) is a canonical product if $f(z) = z^q g(z)$, where

$$g(z) = \prod_{n=1}^{\infty} \left(1 - \left(\frac{z}{\gamma_n} \right) \right) \exp(w_n(z; \gamma))$$

for $w_n(z; \gamma) \equiv \sum_{r=1}^p r^{-1} \left(\frac{z}{\gamma_n} \right)^r$. The genus of $f(z)$ is the smallest non-negative integer p such that the sum

$$\sum_{n=1}^{\infty} |\gamma_n|^{-(1+p)} \quad (7)$$

is finite. Let

$$V_N(r; \gamma) = \sum_{n=1}^N \gamma_n^{-r}.$$

It will be shown that $f(z)$ satisfies the Gauss-Lucas relation (2) if the simple condition

$$\lim_{N \rightarrow \infty} V_N(r; \gamma) = 0 \quad (8)$$

holds for $r \in \{1, \dots, p\}$. Noting that the left-hand side of (8) represents an infinite sum of vectors in C^p (with coordinates indexed by r) and that this sum may depend on the ordering of the non-zero roots γ_n of the canonical product $g(z)$, it is natural to ask if there is some reordering of these roots that makes (8) hold.

The Levy-Steinitz theorem for conditionally convergent series of vectors is relevant to this question. The treatment of the Levy-Steinitz theorem that yields (8) is due to Katznelson and McGehee [6]. These authors consider vectors in R^∞ , so it is useful to write γ_n^{-r} in the form $x_n(r; \gamma) + ix_n(r+p; \gamma)$, for $1 \leq r \leq p$, where $x_n(k; \gamma) \in R$, for $1 \leq k \leq 2p$ and $i = \sqrt{-1}$.

Lemma 1 *Suppose $\gamma = (\gamma_n : n \geq 1)$ is a sequence of complex numbers such that (3) holds. Then there exists a permutation π of the integers $(n \geq 1)$ such that the rearrangement $\delta_n \equiv \gamma_{\pi(n)}$ satisfies (8) if $\delta = (\delta_n : n \geq 1)$ is substituted for γ . (Note the roots stay the same, only their ordering changes.)*

Proof. It follows from Riemann's theorem that for each $k \in \{1, \dots, 2p\}$, the series

$$X(k; \gamma) \equiv \sum_{n=1}^{\infty} x_n(k; \gamma)$$

can be rearranged to sum to any real number, *in particular to the real number 0*. (The rearrangement may depend on k .) Let S_X stand for the subset of R^{2p} defined by $\{(X(k; \delta) : 1 \leq k \leq 2p), \delta \in \Lambda\}$, where Λ consists of all rearrangements $\delta = (\delta_n : n \geq 1)$ of $\gamma = (\gamma_n : n \geq 1)$ that yield a convergent sum for the vector series $\sum_{n=1}^{\infty} X_n(\delta)$ in R^{2p} with summands

$$X_n(\delta) \equiv (x_n(k; \delta) : 1 \leq k \leq 2p) .$$

It follows from Theorem 1 of Katznelson and McGehee that S_X is the solution of a finite number of homogeneous linear equations in R^{2p} . In particular, S_X is a linear subspace of R^{2p} and contains the zero vector. Translating back from R^{2p} to C^p , it follows that (8) is satisfied if γ is replaced by any rearrangement δ corresponding to the zero vector in S_X . \square

Remark 1. The hypothesis of Lemma 1 can be replaced with the statement that the sequence $(\gamma_n^{-r} : n \geq 1)$ has a rearrangement summing to 0, for each $r \in \{1, \dots, p\}$. This modification allows the cases when $Re(\gamma_n^{-r})$ or $Im(\gamma_n^{-r})$ is absolutely summable to 0. Note further that the original hypothesis is consistent with this modification, as follows from Riemann's theorem and the work of Katznelson and McGehee. (Without the latter work, it is not clear that a sequence $x_n + iy_n$ is rearrangeably summable to zero if the same holds separately for x_n and y_n .)

Definition 1. Let $f(z) = z^q g(z)$, where $g(z)$ is a canonical product of genus p . Then $f(z)$ is *rearrangeable* if the (non-zero) roots of $g(z)$ can be rearranged so as to satisfy (8).

Theorem 1 *If the entire function $f(z) = z^q g(z)$ is rearrangeable, then it satisfies the Gauss-Lucas relation (2).*

Proof. To start, assume $q=0$. It is desired to show that

$$g(z) = \lim_{N \rightarrow \infty} f_N(z; \delta) \tag{9}$$

uniformly on compact subsets of C , where

$$f_N(z; \delta) \equiv \prod_{n=1}^N (1 - (z/\delta_n))$$

and $\delta = (\delta_n : n \geq 1)$ is a rearrangement of the roots of $g(z)$ satisfying (8). Let

$$h_N(z; \delta) = \sum_{r=1}^p r^{-1} V_N(r; \delta) z^r .$$

It is shown in Ahlfors [1, p.193] that the partial canonical product

$$g_N(z; \delta) = \left(\prod_{n=1}^N (1 - (z/\delta_n)) \right) \exp(h_N(z; \delta))$$

is absolutely and uniformly convergent to $g(z)$ on compact subsets of C . (Thus $g(z)$ is rearrangement-invariant, i.e. it does not depend on the ordering of its roots.) Note $h_N(z; \delta)$ converges uniformly to 0 on compact subsets of C because the rearrangement δ satisfies (8). Furthermore (9) holds since $|f_N(z; \delta) - g(z)|$ is dominated by

$$|(\exp(-h_N(z; \delta)) - 1)g_N(z)| + |g_N(z) - g(z)|$$

(the triangle inequality for complex numbers).

It remains to show that $f(z)$ satisfies (2). Assume, by way of contradiction, that there exists $z_0 \in C$ with $f'(z_0) = 0$ and $z_0 \notin \bar{H}(f^{(-1)}(0))$. Let K be a closed convex subset of C with $z_0 \notin K$ and $f^{(-1)}(0) \subset K$. By a theorem due to Hurwitz there exists a sequence of complex numbers v_n tending to z_0 and a sequence of positive integers M_n tending to infinity such that $f'_{M_n}(v_n) = 0$ for all n . (See Titchmarsh [9].) It follows (by the Gauss-Lucas theorem for polynomials and the fact that $f_N^{(-1)}(0) \subset f^{(-1)}(0)$ for all N) that $v_n \in K$ for all n . This contradicts the convergence of v_n to z_0 .

The restriction that $q=0$ can be removed because the argument that $f(z)$ satisfies the Gauss-Lucas relation (2) is not affected by multiplication by the monomial z^q . \square

Remark 2. It should be noted that polynomials and all other entire functions of genus zero are rearrangeable because condition (8) is vacuous if $p=0$.

3 Multivariate entire functions

In this section it will be shown that the multivariate Gauss-Lucas relation (4) follows by successively holding constant all variables but one. Hormander [4] is the standard reference about multivariate entire functions (e.g. Hartog's theorem that separate analyticity implies joint analyticity).

Lemma 2 *Suppose $f(z) = f(z_1, \dots, z_M)$ is an entire function such that for all $m \in \{1, \dots, M\}$ and all $z^{(m)} \in C^{M-1}$, the univariate entire function $g(w) \equiv f(w, z^{(m)})$ satisfies the univariate Gauss-Lucas relation (2). Then $f(z)$ satisfies the multivariate Gauss-Lucas relation (4).*

Proof. Suppose $f_{,m}(z) = 0$. For $g(w)$ as above, note that the derivative

$$g'(z_m) = f_{,m}(z_m, z^{(m)}) = f_{,m}(z) .$$

It follows $g'(z_m) = 0$ and thus $z_m \in \bar{H}(g^{(-1)}(0))$. Let K be any closed separately convex subset of C^M containing $f^{(-1)}(0)$. The section $K(z^{(m)})$ is a closed convex subset of C containing $g^{(-1)}(0)$. Thus $K(z^{(m)})$ contains $\bar{H}(g^{(-1)}(0))$. In particular $z_m \in K(z^{(m)})$, i.e. $z \in K$. By choice of K this shows $z \in \bar{H}_2(f^{(-1)}(0))$. \square

Remark 3. If the hypothesis in Lemma 2 is restated as “ $g(w)$ satisfies the Gauss-Lucas relation (1)”, then the conclusion becomes “ $f(z)$ satisfies the multivariate Gauss-Lucas relation (6)”; the proof is the same except that the closures of $H(g^{-1}(0))$ and $H_2(f^{(-1)}(0))$ are not taken. This change reflects the simple argument in Kanter [5] concerning multivariate polynomials.

Theorem 2 *Suppose $f(z_1, \dots, z_M)$ is an entire function such that the univariate function $g(w) = f(w, z^{(m)})$ is rearrangeable for all $m \in \{1, \dots, M\}$ and all $z^{(m)} \in C^{M-1}$. Then $f(z)$ satisfies the Gauss-Lucas relation (4).*

Proof. Apply Theorem 1 and Lemma 2. \square

Theorem 2 can be immediately applied to multivariate stable entire functions.

Definition 2. Let $\theta = (\theta_1, \dots, \theta_M) \in R^M$ and let

$$A(\theta) = \{z \in C^M : \text{Im}(e^{i\theta_m} z_m) > 0 \text{ for } 1 \leq m \leq M\} .$$

A multivariate entire function is called θ -stable if it has no zeros in $A(\theta)$.

Remark 4. Let $A^c(\theta)$ stand for the complement of $A(\theta)$ in C^M . It is easy to see that $A^c(\theta)$ is a closed separately convex subset of C^M . (See [5].)

Corollary 1 *Suppose the multivariate θ -stable entire function $f(z)$ satisfies the hypotheses of Theorem 2. Then any non-null partial derivative $f_{,m}(z)$ is also θ -stable.*

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